

Gamow Vectors in a Periodically Perturbed Quantum System

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Abstract Gamow vectors and resonances play an important role in scattering theory, especially in the physics of metastable states. We study Gamow vectors and resonances in a time-dependent setting using the Borel summation method. In particular, we analyze the behavior of the wave function $\psi(x, t)$ for one dimensional time-dependent Hamiltonian $H = -\partial_x^2 \pm 2\delta(x)(1 + 2r \cos \omega t)$ where $\psi(x, 0)$ is compactly supported.

We show that $\psi(x, t)$ has a Borel summable expansion containing finitely many terms of the form $\sum_{n=-\infty}^{\infty} e^{i^{3/2}\sqrt{-\lambda_k+n\omega}|x|} A_{k,n} e^{-\lambda_k t + n\omega i t}$, where λ_k represents the associated resonance. This expression defines Gamow vectors and resonances in a rigorous and physically relevant way for all frequencies and amplitudes in a time-dependent model.

For small amplitude ($|r| \ll 1$) there is one resonance for generic initial conditions. We calculate the position of the resonance and discuss its physical meaning as related to multiphoton ionization. We give qualitative theoretical results as well as numerical calculations in the general case.

Keywords Gamow vectors · Resonances · Borel summation

1 Introduction

Gamow vectors and resonances, introduced by Gamow to describe α -decay (cf. [1, 25]), are very important mathematical tools in the study of metastable (or quasistable) states in quantum mechanics (cf. [2]). The decay states described by Gamow vectors are also linked to the Fermi-Dirac golden rule (cf. [9]). From a physical point of view, resonances show up as “bumps” in some experimentally measured quantities such as the scattering cross-section (see e.g. [16] and the references therein). Resonances also appear in the study of Stark effect, two-body problem in atomic and molecular physics, etc. [17]. There are numerous definitions of resonances and resonant states, using the scattering matrix, rigged Hilbert spaces, Green’s function, etc. (cf. [9, 10] and the references therein). These definitions rely on the

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time-independent Schrödinger equation, though they may be extended to time-dependent settings in a perturbative regime (cf. [3, 14, 16, 19]). Another less general but very effective method of studying resonances is the complex scaling method [17, 18]. The above various definitions are expected to be equivalent [19], and in certain cases the equivalence has been established [20].

In a recent paper [27], the author and his collaborator gave a rigorous definition of Gamow vectors and resonances for compactly supported time-independent potentials in one dimension, using Borel summation (for a detailed description of Borel summation, see [8, 27]). Resonance expansions in one-dimensional time-dependent fields were also studied in [26], though the results were not very explicit and Borel-summability were not proved. In this paper, we study the resonances associated to a time-dependent periodic one-dimensional delta potential and obtain a complete Borel-summable expansion. In our case, the Gamow vector is of the form of the so-called Floquet ansatz (cf. [4, 26]). Our result holds for all amplitudes and frequencies of the time-dependent field. In the case of small amplitude or high frequency, we calculate the resonances asymptotically, and the real part of the resonances measures the ionization rate. In this sense, our paper extends the results of [5, 15]. As we will see, time dependency introduces new subtleties and complex phenomena.

It is worth mentioning that the simple model we present here not only allows for explicit calculation, but also contains many of the essential ingredients of the ionization process in real systems [5]. For instance, it reproduces many features of the experimental curves for the multiphoton ionization of excited hydrogen atoms by a microwave field.

2 Setting and Main Results

We consider the time-dependent one-dimensional Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x, t) \psi(x, t)$$

where the potential $V(x, t)$ is a delta function potential well or barrier with a time-periodic perturbation. In this paper, we consider two simple but illuminating cases:

- (1) delta potential well $V(x, t) = -2A\delta(x)(1 + 2r \cos \omega t)$,
- (2) delta potential barrier $V(x, t) = 2A\delta(x)(1 + 2r \cos \omega t)$.

Here $A > 0$ represents the strength of the potential, r represents the relative amplitude of the perturbation and ω the frequency. Without loss of generality we take $r > 0$, $\omega > 0$. We further assume the initial wave function $\psi_0(x) := \psi(x, 0)$ is compactly supported and C^2 on its support.

We first normalize the equation by changing variables $x \rightarrow \frac{\hbar}{2mA}x$, $t \rightarrow \frac{\hbar^2}{2mA^2}t$, $\omega \rightarrow \frac{2mA^2}{\hbar^2}\omega$. Note that this is more than using atomic units since we also used the special property of the delta function $\delta(Ax) = \delta(x)/A$. The equation becomes

$$i \frac{\partial}{\partial t} \psi(x, t) = -\frac{\partial^2}{\partial x^2} \psi(x, t) \mp 2\delta(x)(1 + r \cos \omega t) \psi(x, t) \quad (1)$$

(where “−” corresponds to the delta potential well and “+” corresponds to the barrier). We shall focus on the delta potential well and analyze in detail the behavior of the wave function as well as the resonances of the system for all amplitudes and frequencies. The analysis of the delta potential barrier is very similar and we will give the results in Sect. 4 without detailed proofs.

Theorem 2.1 Assume the initial wave function $\psi(x, 0)$ is compactly supported and C^2 on its support, then we have for all $t > 0$

$$\begin{aligned} \psi(x, t) &= \sum_{k=1}^K \sum_{n=-\infty}^{\infty} e^{i^{3/2}\sqrt{-\lambda_k+n\omega i}|x|} A_{k,n} e^{-\lambda_k t+n\omega i t} \\ &\quad - \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{e^{i\theta}\infty} e^{i^{3/2}\sqrt{-q+n\omega i}|x|+n\omega i t-qt} \varphi_n(-q) dq \\ &\quad - \frac{1}{2\pi i} \int_0^{e^{i\theta}\infty} F(x, -q) e^{-qt} dq \end{aligned}$$

where $\lambda_k + n\omega i$ are resonances of the system ($\text{Re}(\lambda_k) > 0$), φ a ramified analytic function with square root branch points at every $n\omega i$ ($n \in \mathbb{Z}$), and F an explicit function with $\sqrt{p}F(p)$ analytic in \sqrt{p} . θ is a small angle chosen to ensure that no resonance lies on the path of integration.

Moreover, the coefficients $A_{k,n}$ satisfy the recurrence relation

$$\left(\sqrt{-i}\sqrt{i+n\omega i-\lambda_k}-1\right)A_{k,n}=rA_{k,n-1}+rA_{k,n+1} \tag{2}$$

and $\psi(x, t)$ has the Borel summable representation

$$\begin{aligned} \psi(x, t) &= \left(i^{3/2}r \int_{-\infty}^{\infty} \psi_0(x) dx\right) t^{-1/2} + \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} C_{n,k}(x) e^{n\omega i t} t^{-3/2-k} \\ &\quad + \sum_{k=1}^K \sum_{n=-\infty}^{\infty} e^{i^{3/2}\sqrt{-\lambda_k+n\omega i}|x|} A_{k,n} e^{-\lambda_k t+n\omega i t}. \end{aligned}$$

Corollary 2.1 For $1 \leq k \leq K$, the Gamow vector term

$$\sum_{n=-\infty}^{\infty} e^{i^{3/2}\sqrt{-\lambda_k+n\omega i}|x|} A_{k,n} e^{-\lambda_k t+n\omega i t}$$

is a generalized eigenvector of the Hamiltonian, in the sense that it solves (1), but grows exponentially (in a prescribed fashion) for large $|x|$.

Proposition 2.2 For small r there is only one array of resonances, i.e. $K = 1$. The asymptotic position of the array of resonances and a similar result for large ω are given in Sect. 3.4.

As $r \in \mathbb{R}$ varies continuously, no resonance can appear or disappear while away from the imaginary line on the non-principal Riemann sheet, and the position of each resonance changes continuously on the Riemann sheet.

In the above formulas the branch of the square root is chosen to be the usual one: $\arg(z) \in (-\pi, \pi]$ and $\arg(\sqrt{z}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. We refer to this choice of branch when we use the phrase ‘‘usual (choice of) branch’’ in this paper.

For small r we calculate asymptotically the position of the resonance, which is related to the ionization rate. For generic r we will give numerical results showing that the Gamow

vector terms exist for some but not all r , and we plot the graph of the positions of resonances with different amplitudes (see Sect. 4).

Remark 2.3 Theorem 2.1 and its corollaries generalize to the case where

$$V(x, t) = \mp 2A\delta(x) \left(1 + 2 \sum_{k=1}^{K_0} (r_k \cos k\omega t + s_k \sin k\omega t) \right).$$

Remark 2.4 Borel summation [8] is a canonical extension of usual summation in the sense that it preserves all properties of usual summation. It provides an exact and unique representation for the wave function, a rigorous definition for Gamow vectors and resonances, as well as practical ways to calculate its asymptotic behavior (e.g. least term truncation, cf. [21, 27]). Though relatively new, Borel summation has become a standard method for interpreting divergent series in the study of various physical systems and phenomena, including the Zeeman effect [17], hydrogen molecular ion [17], and in particular resonances [22, 23]. However, in the existing literatures Borel summation is used only to study perturbation series, while in our paper we obtain an exact Borel-summable expansion in time t for the wave function. Methods based on perturbation theory, such as those in [19] and [16], only uncovers one or a few resonances, while the Borel-summable expansion in this paper gives all resonances.

3 Proof of Main Results

3.1 Integral Reformulation of the Equation

We first consider the Laplace transform in t

$$\hat{\psi}(x, p) = \int_0^\infty e^{-pt} \psi(x, t) dt.$$

The existence of this Laplace transform (for $\text{Re}(p) > 0$) follows from the existence of a strongly differentiable unitary propagator (see Theorem X.71, [6] vol. 2, p. 290, see also [7], [27] and [15]). As we will see, Theorem 1 follows from analyzing the singularities (poles and branch points) of the analytic continuation of $\hat{\psi}(x, p)$.

Performing this Laplace transform on (1), we obtain

$$ip\hat{\psi}(x, p) - i\psi_0(x) = -\frac{\partial^2}{\partial x^2} \hat{\psi}(x, p) - 2\delta(x)\hat{\psi}(x, p) - 2r\delta(x)\hat{\psi}(x, p - i\omega) - 2r\delta(x)\hat{\psi}(x, p + i\omega). \tag{3}$$

We then rewrite the above ordinary differential equation as an integral equation by inverting the operator $\frac{\partial^2}{\partial x^2} + ip$. We have

$$\hat{\psi}(x, p) = \frac{\sqrt{i}e^{-i^{3/2}\sqrt{p}x}}{2\sqrt{p}} \int_{+\infty}^x e^{i^{3/2}\sqrt{p}s} g(s) ds - \frac{\sqrt{i}e^{i^{3/2}\sqrt{p}x}}{2\sqrt{p}} \int_{-\infty}^x e^{-i^{3/2}\sqrt{p}s} g(s) ds$$

where

$$g(x) = i\psi_0(x) - 2\delta(x)\hat{\psi}(x, p) - 2r\delta(x)\hat{\psi}(x, p - i\omega) - 2r\delta(x)\hat{\psi}(x, p + i\omega).$$

Recalling that $\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$, we simplify the above integral equation and obtain

$$\hat{\psi}(x, p) = \frac{e^{-i^{3/2}\sqrt{p}x}}{2i^{-3/2}\sqrt{p}} \int_{+\infty}^x e^{i^{3/2}\sqrt{p}s}\psi_0(s)ds - \frac{e^{i^{3/2}\sqrt{p}x}}{2i^{-3/2}\sqrt{p}} \int_{-\infty}^x e^{-i^{3/2}\sqrt{p}s}\psi_0(s)ds + \frac{\sqrt{i}e^{i^{3/2}\sqrt{p}|x|}}{\sqrt{p}} \left(\hat{\psi}(0, p) + r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) \right). \tag{4}$$

Letting $x = 0$ we get an equation for $\hat{\psi}(0, p)$

$$\hat{\psi}(0, p) = \frac{i^{3/2}}{2\sqrt{p}} \int_{+\infty}^0 e^{i^{3/2}\sqrt{p}s}\psi_0(s)ds - \frac{i^{3/2}}{2\sqrt{p}} \int_{-\infty}^0 e^{-i^{3/2}\sqrt{p}s}\psi_0(s)ds + \frac{\sqrt{i}}{\sqrt{p}} \left(\hat{\psi}(0, p) + r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) \right) \tag{5}$$

which implies

$$\frac{\sqrt{i}}{\sqrt{p}} \left(\hat{\psi}(0, p) + r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) \right) = \hat{\psi}(0, p) - \frac{i^{3/2}}{2\sqrt{p}} \int_{+\infty}^0 e^{i^{3/2}\sqrt{p}s}\psi_0(s)ds - \frac{i^{3/2}}{2\sqrt{p}} \int_{-\infty}^0 e^{-i^{3/2}\sqrt{p}s}\psi_0(s)ds. \tag{6}$$

Substituting (6) in (4) we get

$$\hat{\psi}(x, p) = e^{i^{3/2}\sqrt{p}|x|}\hat{\psi}(0, p) + f(x, p) - e^{i^{3/2}\sqrt{p}|x|}f(0, p) \tag{7}$$

where

$$f(x, p) = \frac{i^{3/2}e^{-i^{3/2}\sqrt{p}x}}{2\sqrt{p}} \int_{+\infty}^x e^{i^{3/2}\sqrt{p}s}\psi_0(s)ds - \frac{i^{3/2}e^{i^{3/2}\sqrt{p}x}}{2\sqrt{p}} \int_{-\infty}^x e^{-i^{3/2}\sqrt{p}s}\psi_0(s)ds.$$

Equation (7) indicates that the analytic continuation of $\hat{\psi}(x, p)$, as well as its singularities, follows naturally from that of $\hat{\psi}(0, p)$, so it suffices to analyze $\hat{\psi}(0, p)$ using the recurrence relation (5). Later we will perform the inverse Laplace transform on $\hat{\psi}(x, p)$, justified by estimating $\hat{\psi}(0, p)$ and $f(x, p)$ for large p . We will then deform the contour of the Bromwich integral, which yields the expression in Theorem 2.1. It is worth noting that to deform the contour it suffices to place a branch cut of the square root in the left half complex plane, while to analyze the singularities of $\hat{\psi}(0, p)$ we need to consider a larger region in the Riemann surface. Some delicate points of the analysis stem from the complexity of the Riemann surface, since, as we will see, $\hat{\psi}(0, p)$ has infinitely many branch points and there appears to be a barrier of singularities on the non-principal Riemann sheet.

3.2 Recurrence Relation and Analyticity of $\hat{\psi}$

We rewrite the recurrence relation (5) as

$$\left(\sqrt{-i}\sqrt{p} - 1 \right) \hat{\psi}(0, p) = r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) + \sqrt{-i}\sqrt{p}f(0, p).$$

We will show that $f(0, p) = \frac{\psi_0(0)}{p} + O(\frac{1}{p^{3/2}})$ as $p \rightarrow \infty$ in any direction in the right half complex plane (see Sect. 3.6). It is not *a priori* clear that $\hat{\psi}(0, p)$ has an inverse Laplace transform. We thus let $\tilde{\psi}(p) = \hat{\psi}(0, p) - f(0, p)$. The recurrence relation for $\tilde{\psi}$ is

$$(\sqrt{-i}\sqrt{p} - 1) \tilde{\psi}(p) = r\tilde{\psi}(p - i\omega) + r\tilde{\psi}(p + i\omega) + (1 + 2r)f(0, p). \tag{8}$$

It is convenient to write the recurrence relation in a difference equation form. Denoting $p = i + i\omega + z$, $y_n(z) = \tilde{\psi}(i + i\omega + z)$, and $f_n(z) = (1 + 2r)f(0, i + i\omega + z)$, we have

$$(\sqrt{-i}\sqrt{i + i\omega + z} - 1) y_n(z) = ry_{n-1}(z) + ry_{n+1}(z) + f_n(z). \tag{9}$$

The associated homogeneous equation is of course

$$(\sqrt{-i}\sqrt{i + i\omega + z} - 1) y_n(z) = ry_{n-1}(z) + ry_{n+1}(z). \tag{10}$$

Let z_0 be a branch point closest to 0, that is, a point on the imaginary axis satisfying $-z_0i = \inf_n \{|1 + n\omega|\}$ (note that $|z_0| \leq \frac{\omega}{2}$), and let n_0 be the corresponding n . Since clearly $y_n(z) = y_{n+1}(z - i\omega) = y_{n-1}(z + i\omega)$, it suffices to consider $\text{Im}(z) \in (-\frac{4}{5}\omega, \frac{4}{5}\omega)$ for the usual branch. In general, if we make a branch cut at $(e^{i\theta}\infty, z_0)$ ($\cos\theta \neq 0$) we consider the strip-shaped region $\{|\text{Im}(z) - \rho \sin\theta| < \frac{4}{5}\omega, \text{Re}(z) = \rho \cos\theta, \rho \in \mathbb{R}\}$.

To analytically continue $y := \{y_n\}$, we consider the Hilbert space \mathcal{H} defined by

$$\|\mathbf{x}\|_{\mathcal{H}}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2})|x_n|^2$$

and the operator $C_m : \mathcal{H} \rightarrow \mathcal{H}$

$$(C_m \mathbf{y})_n(z) = \frac{(1 + m\sqrt{i})y_n(z) + ry_{n-1}(z) + ry_{n+1}(z)}{(\sqrt{-i}\sqrt{i + i\omega + z} + m\sqrt{i})} \quad (m \in \mathbb{Z}^+).$$

It is easy to see that C_m is entire in r and analytic in $\sqrt{z - z_0}$ in the region $\text{Re}(z) > -m^2, \text{Im}(z) \in (-\frac{4}{5}\omega, \frac{4}{5}\omega)$.

Lemma 3.1 C_m is a compact operator for any choice of branch.

Proof For arbitrarily large $N \in \mathbb{N}$, we consider the finite rank operator $\mathcal{D}_{m,N} : \mathcal{H} \rightarrow \mathcal{H}$

$$(\mathcal{D}_{m,N} \mathbf{y})_n = \begin{cases} (C_m \mathbf{y})_n, & |n| < N \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that

$$\|C_m - \mathcal{D}_{m,N}\| = O(N^{-1/2}).$$

Therefore C_m , being the limit of finite rank operators in operator norm, is compact. □

Lemma 3.2 *The equation*

$$(\sqrt{-i}\sqrt{i + i\omega + z} - 1) y_n(z) = ry_{n-1}(z) + ry_{n+1}(z) + g_n(z)$$

has a unique solution in \mathcal{H} for $|\operatorname{Re}(z)| > (2r + 1)^2$, for all $\mathbf{g} \in \mathcal{H}$. In particular, (9) has a unique solution and (10) has only the trivial solution $y = 0$. The conclusion holds as well if $z \neq 0$ and r is sufficiently small. Furthermore, for large $|\operatorname{Re}(z)|$ we have $|\mathbf{y}| = O(|\operatorname{Re}(z)|^{-1/2}|\mathbf{g}|)$ where $|\mathbf{x}| := \sup_n |x_n|$.

Proof Note that under the assumptions above, the norm of the linear operator $S : \mathcal{H} \rightarrow \mathcal{H}$

$$(S\mathbf{y})_n(z) = \frac{ry_{n-1}(z) + ry_{n+1}(z)}{(\sqrt{-i}\sqrt{i + in\omega + z} - 1)}$$

is smaller than 1, since $|\sqrt{-i}\sqrt{i + in\omega + z} - 1| \geq |\sqrt{i + in\omega + z}| - 1 \geq \sqrt{|\operatorname{Re}(z)|} - 1 > 2r$. We then have

$$\mathbf{y} = \frac{(\mathcal{I} - S)^{-1}\mathbf{g}}{(\sqrt{-i}\sqrt{i + in\omega + z} - 1)}. \quad \square$$

Proposition 3.3 *For every $r \in \mathbb{C}$, there are at most finitely many $z = z_1, \dots, z_{l_r}$ for which the homogeneous equation (10) has a nonzero solution \mathbf{y} in \mathcal{H} . For all other z , there exists a unique solution to (9). The function $\sqrt{z - z_0}\mathbf{y}$ is analytic in both $\sqrt{z - z_0}$ and r , and it can be analytically continued on the Riemann surface of $\sqrt{i + in\omega + z}$ to $\arg z \in (-3\pi/2, 3\pi/2)$. (in other words, one can rotate the branch cut in the left half complex plane). Moreover, z_1, \dots, z_{l_r} are either poles (in $\sqrt{z - z_0}$) or removable singularities of \mathbf{y} , and $y_n (n \neq n_0)$ is analytic in $\sqrt{z - z_0}$ when z is close to z_0 .*

Proof We consider the equation

$$\mathbf{y}^{[m]} = C_m \mathbf{y}^{[m]} + \frac{1}{(\sqrt{-i}\sqrt{i + in\omega + z + m\sqrt{i}})} \mathbf{f}.$$

Since C_m is compact, analytic in both r and $\sqrt{z - z_0}$, and invertible for $|\operatorname{Re}(z)| > (2r + 1)^2$, it follows from the analytic Fredholm alternative (see [6] vol. 1, Theorem VI.14, p. 201) that the proposition is true for every $\mathbf{y}^{[m]}$ (note that the solution of the inhomogeneous equation exists for $|\operatorname{Re}(z)| > (2r + 1)^2$, thus there can only be finitely many isolated singularities). Uniqueness of the solution implies $\mathbf{y}^{[m]} = \mathbf{y}^{[m+1]}$ for all $r \in \mathbb{C}, \operatorname{Re}(z) > -m^2$. Thus we naturally define the analytic continuation of the solution to be $\mathbf{y} := \mathbf{y}^{[m]}$. Analytic continuation on the Riemann surface follows from the fact that for fixed r, z (z not on the branch cut) slightly rotating the branch cut does not change the value of $\sqrt{i + in\omega + z}$ for any $n \in \mathbb{Z}$. Uniqueness of the solution thus ensures \mathbf{y} also remains unchanged.

Assume $y_n(z) \sim b_n(z - z_0)^{-1/2}$ as $z \rightarrow z_0$. It is easy to see from (9) that

$$\begin{aligned} (\sqrt{-i}\sqrt{i + in\omega + z} - 1) b_n &= r b_{n-1} + r b_{n+1} \quad (n \neq n_0), \\ (\sqrt{-i}\sqrt{i + in_0\omega + z_0} - 1) b_{n_0} &= r b_{n_0-1} + r b_{n_0+1} - (1/2 + r) i^{3/2} \int_{-\infty}^{\infty} \psi_0(x) dx. \end{aligned}$$

The unique solution of this recurrence relation is obviously

$$b_{n_0} = (1/2 + r) i^{3/2} \int_{-\infty}^{\infty} \psi_0(x) dx, \quad b_n = 0 \quad (n \neq n_0). \quad \square$$

Corollary 3.4 *For every $r \in \mathbb{C}$, (8) has a unique solution $\tilde{\psi}$. $\sqrt{p}\tilde{\psi}$ is meromorphic in p with square root branches at every $in\omega$ ($n \in \mathbb{Z}$) and poles at $\{p_k + in\omega\}$ ($k = 1, 2, \dots, l_r, n \in \mathbb{Z}$).*

Proof In order to recover $p = i + i\omega + z$ from the solution to (9), we only need to show $y_n(z) = y_{n\mp 1}(z \pm \omega i)$. To this end, note that by (9) we have

$$\begin{aligned} & \left(\sqrt{-i}\sqrt{i + i\omega + z} - 1\right)y_{n\mp 1}(z \pm \omega i) \\ &= ry_{n\mp 1-1}(z \pm \omega i) + ry_{n\mp 1+1}(z \pm \omega i) + f_{n\mp 1}(z \pm \omega i) \end{aligned} \tag{11}$$

which is the same equation as (9) since $f_{n\mp 1}(z \pm \omega i) = f_n$.

Thus, uniqueness of the solution (Proposition 3.3) implies $y_n(z) = y_{n\mp 1}(z \pm \omega i)$. Note that we need to choose the same branch for all $\sqrt{i + i\omega + z}$. \square

We conclude this section with a few observations about the positions of the poles of $\tilde{\psi}$, including the well-known result of complete ionization (see [5, 7, 15]).

Proposition 3.5 *For $r > 0$, \mathbf{y} has no pole on the imaginary axis or the right half complex plane, with the usual choice of branch.*

Proof In view of Proposition 3.3, we only need to show the homogeneous equation (10) has no nonzero solution in \mathcal{H} . Multiplying (10) by $\bar{y}_n(z)$ and summing in n we get

$$\sum_{n=-\infty}^{\infty} \left(\sqrt{-i}\sqrt{i + i\omega + z} - 1\right) |y_n|^2 = 2r \sum_{n=-\infty}^{\infty} \operatorname{Re}(y_{n-1}\bar{y}_n)$$

which implies

$$\sum_{n=-\infty}^{\infty} \sqrt{-i}\sqrt{i + i\omega + z} |y_n|^2$$

must be real.

If $\operatorname{Re}(z) \geq 0$ then $\operatorname{Im}(\sqrt{-i}\sqrt{i + i\omega + z}) \leq 0$ for all n and $\operatorname{Im}(\sqrt{-i}\sqrt{i + i\omega + z}) < 0$ for all $n < -(1 + |z|)/\omega$. Thus $y_n = 0$ for all $n < -(1 + |z|)/\omega$ and (10) implies $\mathbf{y} = 0$. \square

Proposition 3.6 *For $r > 0$, \mathbf{y} has no pole on the imaginary axis for any choice of branch.*

Proof Similar to the above. Note that $\operatorname{Re}(z) = 0$ implies $\operatorname{Im}(\sqrt{-i}\sqrt{i + i\omega + z}) = 0$ for all $n > -(1 + \operatorname{Im}(z))/\omega$ and $\operatorname{Im}(\sqrt{-i}\sqrt{i + i\omega + z})$ has the same sign (and nonzero) for all $n < -(1 + \operatorname{Im}(z))/\omega$. \square

Proposition 3.7 *Solutions of the homogeneous equation (10) exist in negative conjugate pairs, in the sense that if z_1 is a pole of ψ , then $-\bar{z}_1$ is also a pole (with a different choice of branch, see proof and comments below).*

Proof Simply note that $(-i)^{1/2}\sqrt{i + i\omega + z} = \overline{(-i)^{1/2}\sqrt{i + i\omega - \bar{z}}}$ if we choose the branches in such a way that in the upper half complex plane the two square roots are the same, while in the lower half plane they are opposite. \square

In view of the above propositions, we will concentrate our study of resonances on the left half complex plane. The author believes that the imaginary line on the non-principal Riemann surface is a singularity barrier, and the Proposition 3.7 provides a pseudo-analytic continuation across the barrier. We will not discuss the details in this paper.

3.3 The Homogeneous Equation

As we mentioned in the introduction, poles of \mathbf{y} in the left half complex plane correspond to resonances of the system. According to Proposition 3.3, finding these poles is essentially the same as finding solutions to the homogeneous equation (10) in \mathcal{H} .

Lemma 3.8 *Assume the nonzero vector $\mathbf{u} = \{u_n\}$ satisfies the homogeneous recurrence relation (10), and that*

$$\sum_{n=0}^{\infty} (1 + |n|^{3/2}) |u_n|^2 < \infty.$$

Assume also that the nonzero vector $\mathbf{v} = \{v_n\}$ satisfies (10) and

$$\sum_{n=-\infty}^0 (1 + |n|^{3/2}) |v_n|^2 < \infty.$$

Then the homogeneous equation (10) has a nonzero solution in \mathcal{H} if and only if the discrete Wronskian $W := u_n v_{n+1} - v_n u_{n+1} = 0$. The solution, if it exists, is a constant multiple of \mathbf{u} (or equivalently \mathbf{v}).

Proof If $r = 0$ the lemma is trivial. Assume $r > 0$. We first note that the recurrence relation (10) implies

(1) W is independent of n .

(Note that $(u_n v_{n+1} - v_n u_{n+1}) - (u_{n+1} v_{n+2} - v_{n+1} u_{n+2}) = v_{n+1}(u_n + u_{n+2}) - u_{n+1}(v_n + v_{n+2}) = (\sqrt{-i}\sqrt{i + i n \omega + z} - 1)/r(v_{n+1} u_{n+1} - u_{n+1} v_{n+1}) = 0$.)

(2) For any n and any nonzero vector \mathbf{x} satisfying that recurrence relation, we have $|x_n|^2 + |x_{n+1}|^2 \neq 0$, $|x_n|^2 + |x_{n+2}|^2 \neq 0$ ($n \neq -1$).

Now assume $W = 0$. Since \mathbf{v} is nonzero, there exists m for which $v_m \neq 0$. Thus we have $u_{m\pm 1} = (u_m/v_m)v_{m\pm 1}$. Since $\mathbf{u} \neq 0$ we must have $u_m \neq 0$, for otherwise $u_{m\pm 1} = u_m = 0$. If $v_{m\pm 1} = 0$ then $u_{m\pm 1} = 0$, which implies $|u_m - (u_m/v_m)v_m|^2 + |u_{m\pm 1} - (u_m/v_m)v_{m\pm 1}|^2 = 0$, meaning $\mathbf{u} = (u_m/v_m)\mathbf{v}$. If $v_{m\pm 1} \neq 0$ then $u_{m\pm 1} \neq 0$, which inductively implies again $\mathbf{u} = (u_m/v_m)\mathbf{v}$. Therefore \mathbf{u} solves (10) in \mathcal{H} .

If $W \neq 0$ then clearly \mathbf{u} and \mathbf{v} are the two linearly independent solutions of the second order difference equation (10). Furthermore, we have $\liminf_{n < 0} |u_n| > 0$ and $\liminf_{n > 0} |v_n| > 0$, since $\limsup_{n > 0} |u_n| < \text{const.} |n|^{-3/4}$ and $\limsup_{n < 0} |v_n| < \text{const.} |n|^{-3/4}$ but $u_n v_{n+1} - v_n u_{n+1}$ is a nonzero constant. Therefore no nonzero linear combination of \mathbf{u} and \mathbf{v} can be in \mathcal{H} . Since a second order difference equation cannot have any other solution, there is no nonzero solution of (10) in \mathcal{H} . □

We now give a constructive description of \mathbf{u} and \mathbf{v} . For convenience let $h_n(z) = (\sqrt{-i}\sqrt{i + i n \omega + z} - 1)$. We choose $n_{1,2} \in \mathbb{Z}$ so that $|h_n| > 2|r|$ for all $n \geq n_1 > 0$ and $n \leq n_2 < 0$. Let \mathcal{I} be the identity operator. We define $\mathcal{H}_{1,2}$ by

$$\|\mathbf{x}\|_1^2 = \sum_{n=n_1}^{\infty} (1 + |n|^{3/2}) |x_n|^2,$$

$$\|\mathbf{x}\|_2^2 = \sum_{n=-\infty}^{n_2} (1 + |n|^{3/2}) |x_n|^2.$$

Proposition 3.9 *There exist \mathbf{u} and \mathbf{v} , analytic in r and ramified analytic in z , satisfying the conditions described in Lemma 3.8. Moreover, $\mathbf{u}(z \pm \omega i) = \text{const.}\mathbf{u}(z)$ and $\mathbf{v}(z \pm \omega i) = \text{const.}\mathbf{v}(z)$.*

Proof Let $\mathcal{T}_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$

$$(\mathcal{T}_1 \mathbf{y})_n = \begin{cases} \frac{r}{h_n}(y_{n-1} + y_{n+1}), & n > n_1, \\ \frac{r}{h_n}y_{n+1}, & n = n_1. \end{cases}$$

Let $\mathbf{a} = (r/h_{n_1}, 0, 0, \dots) \in \mathcal{H}_1$.

The equation

$$\mathbf{u} = \mathcal{T}_1 \mathbf{u} + \mathbf{a}$$

has a unique solution

$$\mathbf{u} = (\mathcal{I} - \mathcal{T}_1)^{-1} \mathbf{a} = \mathbf{a} + \mathcal{T}_1 \mathbf{a} + \mathcal{T}_1^2 \mathbf{a} \dots$$

since clearly $\|\mathcal{T}_1\| < 1$. It is easy to see that \mathbf{u} satisfies

$$u_n = \begin{cases} \frac{r}{h_n}(u_{n-1} + u_{n+1}), & n > n_1, \\ \frac{r}{h_n}(u_{n+1} + 1), & n = n_1. \end{cases}$$

Thus the recurrence relation

$$u_n = \frac{h_n}{r} u_{n+1} - u_{n+2}$$

extends \mathbf{u} to a solution of the homogeneous equation (10). In particular $u_{n_1-1} = 1$. This solution \mathbf{u} is analytic in r and z (ramified) locally since \mathcal{T}_1 and h_n are analytic in r and z (ramified), and the uniform limit of analytic functions is analytic. As r or $|\text{Im}(z)|$ increases we may analytically continue \mathbf{u} by considering some $n_3 > n_1$ so that $|h_n| > 2|r|$ for all $n \geq n_3$. Using the same procedure as we did for n_1 we get $\tilde{\mathbf{u}}$. It is easy to see that $\mathbf{u} = u_{n_3} \tilde{\mathbf{u}}$ for they both satisfy the contractive recurrence relation (in the sup norm)

$$u_n = \begin{cases} \frac{r}{h_n}(u_{n-1} + u_{n+1}), & n > n_3, \\ \frac{r}{h_n}(u_{n+1} + u_{n_3}), & n = n_3. \end{cases}$$

Note that this implies $u_n \neq 0$ for large n .

The analytic continuation of \mathbf{u} is, up to a scalar multiple, periodic in z . Note that $\mathbf{u}^\pm(z) = \mathbf{u}(z \pm \omega i)$ satisfies (for large $n_3 > n_1$)

$$u_n^\pm = \begin{cases} \frac{r}{h_{n\pm 1}}(u_{n-1}^\pm + u_{n+1}^\pm), & n > n_1, \\ \frac{r}{h_{n\pm 1}}(u_{n+1}^\pm + u_{n_3}^\pm), & n = n_3 \end{cases}$$

while \mathbf{u} satisfies

$$u_{n\pm 1} = \begin{cases} \frac{r}{h_{n\pm 1}}(u_{n\pm 1-1} + u_{n\pm 1+1}), & n > n_3, \\ \frac{r}{h_{n\pm 1}}(u_{n\pm 1+1} + u_{n_3\pm 1}), & n = n_3. \end{cases}$$

Thus $\mathbf{u}(z \pm \omega i) = \frac{u_{n_3}(z \pm \omega i)}{u_{n_3 \pm 1}(z)} \mathbf{u}(z)$.

The construction of \mathbf{v} is very similar, namely $\mathbf{v} = (\mathcal{I} - \mathcal{T}_2)^{-1} \mathbf{b}$ where $\mathcal{T}_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$

$$(\mathcal{T}_2 y)_n \rightarrow \begin{cases} \frac{r}{h_n} (y_{n-1} + y_{n+1}), & n < n_2, \\ \frac{r}{h_n} y_{n-1}, & n = n_2 \end{cases}$$

and $\mathbf{b} = (\dots, 0, 0, r/h_{n_2})$. □

Proposition 3.10 *W is analytic in r and ramified analytic in z. Moreover, W(z) = 0 if and only if W(z ± ωi) = 0.*

Proof The first part is obvious. The second part follows from the relation $\mathbf{u}(z \pm \omega i) = \frac{u_{n_3}(z \pm \omega i)}{u_{n_3 \pm 1}(z)} \mathbf{u}(z)$ (see the proof of the previous proposition) and the fact that $u_{n_3} \neq 0$. □

Remark 3.11 Another way of constructing \mathbf{u} and \mathbf{v} is by using continued fractions, see [5]. The continued fraction expression is slightly simpler in this particular case, but our iteration method can be easily generalized to trigonometric polynomial potentials mentioned in Sect. 2.

3.4 Resonance for Small r

We assume $r > 0$ and analyze the resonances of the system for small r (relative to ω) by locating zeros of W , in view of Lemma 3.8. Since we will need to consider different branch choices, we write for convenience $h_n(z) = ((-i)^{1/2} \sqrt{i + in\omega + z} - 1)$ where the power 1/2 always indicates the usual choice of branch.

Lemma 3.12 *For every choice of branch, there exists a constant c so that when $\omega > c(r + r^2)$, we have $|h_n| > 2r$ for all $n \neq 0$.*

Proof Recall that for a branch cut at $(e^{i\theta} \infty, z_0)$ ($\cos \theta \neq 0$), we consider the strip-shaped region $\Omega_b := \{|\text{Im}(z) - \rho \sin \theta| < \frac{4}{5}\omega, \text{Re}(z) = \rho \cos \theta, \rho \in \mathbb{R}\}$. It is easy to see that $c_1 := \inf_{n \neq 0, z \in \Omega_b} |\frac{z}{\omega} - in| > 0$. Therefore $|h_n(z)| = |(-i)^{1/2} \sqrt{i + in\omega + z} - 1| = \frac{|in\omega + z|}{|\sqrt{i + in\omega + z} + \sqrt{i}|} \geq \frac{|in\omega + z|}{\sqrt{|in\omega + z|} + 2} \geq \frac{c_1 \omega}{\sqrt{c_1 \omega} + 2} > 2r$ if $\sqrt{c_1 \omega} > 2r + 2\sqrt{r}$. Note that $\frac{x^2}{x+2}$ is an increasing function for $x > 0$. □

Proposition 3.13 *For small r, there is a unique nonzero solution of the homogeneous equation (10) in the left half complex plane with the usual choice of branch. Moreover, the solution satisfies*

$$z = \left(\frac{2i}{(1 + \omega)^{1/2} - 1} - \frac{2i}{i^{-1/2} \sqrt{(1 - \omega)i} - 1} + \sigma(r) \right) r^2$$

where $\sigma(r)$ is analytic in r and $\sigma(0) = 0$.

Proof We choose $n_1 = 1, n_2 = -1$ to construct \mathbf{u} and \mathbf{v} . Thus $u_0 = v_0 = 1$ and $W = v_1 - u_1$. We calculate by iterations

$$u_1 = \frac{r}{h_1} + \frac{r^3}{h_1^2 h_2} + \frac{r^5}{h_1^5} R_1,$$

$$v_1 = \frac{h_0}{r} - v_{-1} = \frac{h_0}{r} - \frac{r}{h_{-1}} - \frac{r^3}{h_{-1}^2 h_{-2}} - \frac{r^5}{h_1^5} R_2,$$

$$W = \frac{h_0}{r} - \frac{r}{h_{-1}} - \frac{r}{h_1} - \frac{r^3}{h_1^2 h_2} - \frac{r^3}{h_{-1}^2 h_{-2}} - \frac{r^5}{h_1^5} R_1 - \frac{r^5}{h_1^5} R_2$$

where $R_{1,2}$ are bounded for $\omega > c(r + r^2)$. Note that $|h_0(z)| = |\sqrt{i+z} - i^{1/2}| \geq |z|/2$ and $|\frac{r}{h_n}| \leq \frac{\sqrt{c_1\omega+2}}{c_1\omega} r$ for all $n \neq 0$.

Now, if ω is fixed and r is small, $W = 0$ implies $h_0(z) = O(r^2)$. Hence we must have $z = O(r^2)$. In addition, we need to make the choice of branch so that \sqrt{i} is in the first quadrant. Thus we let $z = (a_0 + \sigma)r^2$ where $\sigma = o(1)$, and we see that

$$\frac{W}{r} = \left(\frac{a_0}{2i} - \frac{1}{(1 + \omega)^{1/2} - 1} - \frac{i^{1/2}}{\sqrt{(1 - \omega)i} - i^{1/2}} \right) (1 + o(1)).$$

Thus we have

$$a_0 = \frac{2i}{(1 + \omega)^{1/2} - 1} - \frac{2i}{i^{-1/2}\sqrt{(1 - \omega)i} - 1}.$$

For small r , W is clearly analytic in both r and σ . Since the value of W depends only on $\bigcup_n \{z : |z - in\omega| < 2a_0r^2\}$, there are exactly two different W with different choices of branch, namely $W_1 : \text{Re}(\sqrt{i}) > 0, \text{Re}(\sqrt{-i}) > 0$ and $W_2 : \text{Re}(\sqrt{i}) > 0, \text{Re}(\sqrt{-i}) < 0$. However, according to Proposition 3.6 and Proposition 3.7, they are in fact negative conjugates to each other, and only one will be in the left half complex plane. We thus take $W = W_1$ for its branch is consistent with the usual branch.

It is easy to verify that

$$\left. \frac{W}{r} \right|_{r=0, \sigma=0} = 0,$$

$$\left. \frac{\partial}{\partial \sigma} \left(\frac{W}{r} \right) \right|_{r=0, \sigma=0} = -\frac{i}{2} \neq 0.$$

Therefore it follows from the implicit function theorem that the position of the zero of W is given by

$$z = \left(\frac{2i}{(1 + \omega)^{1/2} - 1} - \frac{2i}{i^{-1/2}\sqrt{(1 - \omega)i} - 1} + \sigma(r) \right) r^2$$

where $\sigma(r)$ is analytic in r and $\sigma(0) = 0$.

$\sigma(r)$ can be found asymptotically by iterating $\sigma(r) - \frac{2iW}{r}$ as in the standard proof of the implicit function theorem.

Since the usual choice of branch is consistent with W , the zero of W is visible. □

Corollary 3.14 *For r small and $\omega > 1$, the position of the resonance satisfies $\lambda_1 \sim -\frac{2\sqrt{\omega-1}}{\omega} r^2 - \frac{2\sqrt{\omega+1}}{\omega} r^2 i$.*

Proof The corollary follows from the expression of a_0 with the usual choice of branch. The fact that it is indeed a resonance, i.e. a pole of \mathbf{y} , will be established in the next subsection. \square

Remark 3.15 In the case $\omega \gg 1 + r^2$, an analogous analysis shows that the position of the resonance is given by $\lambda_1 \sim -\frac{2r^2}{\sqrt{\omega}} - \frac{2r^2i}{\sqrt{\omega}}$.

Proposition 3.16 *For small r the poles (in one vertical array) of $\tilde{\psi}$ are simple and the residues are nonzero for generic \mathbf{f} .*

Proof We note that the order of the pole of $(\mathcal{I} - C_m)^{-1}$ equals the order of the corresponding zero of $\mathcal{I} - C_m$, which is a constant by the argument principle (see Lemma 3.18 below), since $\mathcal{I} - C_m$ is analytic in z . It is easy to verify that when $r = 0$ the zero of $\mathcal{I} - C_m$ is of order one. Thus the poles are simple.

Let $z = G(r)$ be the continuous functions satisfying $W(G(r), r) = 0, G(0) = 0$. We consider the residue

$$P(r) = \frac{1}{2\pi i} \oint_{|\zeta - G(r)| = \epsilon} y_0(\zeta, r) d\zeta.$$

Obviously $P(r)$ is analytic in r . For generic \mathbf{f} , $P(0) \neq 0$ (in which case \mathbf{y} can be found explicitly). Thus $P(r) \neq 0$ for small r . \square

3.5 Resonances in General

Having analyzed the zeros of W for small r , we proceed to consider the case for general r , as well as the poles of \mathbf{y} .

For convenience we study the region $\Omega_{\theta, \epsilon} := \{z : \text{Im}(z) \in [\rho \sin \theta + \frac{\omega}{2} - \frac{1}{2}\omega + \epsilon, \rho \sin \theta + \frac{\omega}{2} + \frac{1}{2}\omega + \epsilon), \text{Re}(z) = \rho \cos \theta, \rho \in \mathbb{R}\} \cap \{z : |\text{Re}(z)| < (2|r| + 2)^2\}$, the branch cut being placed at $(e^{i\theta} \infty, z_0)$ ($\cos \theta \neq 0$). It is easy to see that there is exactly one zero and one branch point inside this region for small r (cf. Sect. 3.4). We note that as long as z is not located on a branch cut, we may rotate the cut slightly without changing W .

Lemma 3.17 *For every r , W has finitely many zeros in $\bigcup_{|\cos \theta| > c_b > 0} \Omega_{\theta, \epsilon}$ where c_b is arbitrary.*

Proof By Lemma 3.2, there is no zero for $|\text{Re}(z)| > (2|r| + 1)^2$ and the zeros are isolated. Since the Riemann surface of the square root has only two sheets and the region $\Omega_{\theta, \epsilon}$ is bounded, W can only have finitely many zeros. \square

Lemma 3.18 *Assume for some r_0 and arbitrarily small $\epsilon > 0$, with the branch choice $\arg(z) \in (\theta - \epsilon, \theta + 2\pi + \epsilon)$ ($-2\pi < \theta \leq 2\pi, \cos \theta \neq 0$), W has finitely many zeros in $\Omega_{\theta, \epsilon}$. Then the number of zeros remains a constant if r is close to r_0 . Furthermore, each zero moves continuously with respect to r .*

Proof The lemma follows from standard complex analysis arguments. Suppose $W(z, r_0)$ has zeros z_1, z_2, \dots, z_m inside Ω_0 and $\tilde{z}_{m+1}, \dots, \tilde{z}_{m+l}$ on $\partial\Omega_0$. Since $z \in \Omega_{\theta, \epsilon} - \Omega_{\theta, 0}$ iff $z - i\omega \in \Omega_{\theta, 0} - \Omega_{\theta, \epsilon}$, we let $z_{m+k} = \tilde{z}_{m+k} + i\omega$ ($1 \leq k \leq l$). We may choose small $\epsilon > 0$ so that $W(z, r_0)$ has zeros z_1, z_2, \dots, z_{m+l} in $\Omega_{\theta, \epsilon}$ for $\arg(z) \in (\theta - \epsilon, \theta + 2\pi + \epsilon)$, and no other zero in $\Omega_{\theta, 2\epsilon}$ for $\arg(z) \in (\theta - 2\epsilon, \theta + 2\pi + 2\epsilon)$. Let $0 < \delta < \epsilon$ be small so that there

is at most one zero or branch point inside any circle of radius 2δ , and $W(z, r_0)$ is analytic (with a suitable choice of branch) in $|z - z_n| < 2\delta$. Since W is analytic in both z and r , it follows from the argument principle that for r very close to r_0

$$M_n(r) = \frac{1}{2\pi i} \oint_{|\zeta - z_n| = \delta} \frac{\frac{\partial}{\partial \zeta} W(\zeta, r)}{W(\zeta, r)} d\zeta = 1.$$

Now we consider the compact region $\Omega' := \{z : \arg(z) \in [\theta - \epsilon, \theta + 2\pi + \epsilon]\} \cap \overline{\Omega_{\theta, \epsilon}} \setminus \bigcup_{n=1}^m \{z : |z - z_n| < \delta\}$. Clearly $|W(z, r_0)| > 0$ for all $z \in \Omega'$. Since W is jointly uniformly continuous in z and r , we have $|W(z, r)| > 0$ for all $z \in \Omega', r$ close to r_0 .

Thus the number of zeros is locally a constant and they move continuously with respect to r . □

Proposition 3.19 *For every r there are finitely many zeros of W in any strip $\{z : \text{Im}(z) \in [\tilde{z}, \tilde{z} + \omega], \text{Re}(z) \in \mathbb{R}\}$ for all choices of branch within $|\cos \theta| > c_b > 0$, and the position of each zero changes continuously with respect to r .*

Proof The conclusion follows from Proposition 3.10, Lemma 3.17 and 3.18. Note that we may choose θ arbitrarily, thus covering the whole Riemann surface (except for the imaginary lines). □

Remark 3.20 Since W is analytic in r and ramified analytic in z , we expect the zeros of W to be analytic or ramified analytic in r , in view of the Weierstrass preparation theorem in several complex variables (see, for instance, [24] for details). However, we omit the proof in this paper.

As we have shown in Proposition 3.3 and Lemma 3.8, all poles of \mathbf{y} are located where $W = 0$. We summarize the results as

Proposition 3.21 *For generic r and \mathbf{f} , $\mathbf{y}(z, r)$ has finitely many arrays of poles for any choice of branch with $|\cos \theta| > c_b > 0$. Their residues $A_{k,n}$ satisfy the recurrence relation*

$$\left((-i)^{1/2} \sqrt{i + n\omega i - \lambda_1} - 1 \right) A_{k,n} = r A_{k,n-1} + r A_{k,n+1}$$

and $\mathbf{A}_k \in \mathcal{H}$.

Proof The first part is simply a rephrasing of previous results (cf. Proposition 3.19). The recurrence relation for residues follows from the fact that

$$A_{k,n} = \frac{1}{2\pi i} \oint_{|\zeta - G(r)| = \epsilon} y_{k,n}(\zeta, r) d\zeta$$

satisfies the homogeneous equation (10) since \mathbf{y} satisfies (9) and

$$\oint_{|\zeta - G(r)| = \epsilon} f_n(\zeta, r) d\zeta = 0.$$

The above expression for P_n also implies $\mathbf{A}_k \in \mathcal{H}$ since, by Hölder’s inequality

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) |A_{k,n}|^2 \\ & \leq \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) \oint_{|\zeta - G(r)| = \epsilon} |y_n(\zeta, r)|^2 d|\zeta| \\ & = \oint_{|\zeta - G(r)| = \epsilon} \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) |y_n(\zeta, r)|^2 d|\zeta| \leq \sup_{|\zeta - G_1(r)| = \epsilon} \|y(\zeta, r)\|^2 < \infty \end{aligned}$$

the last inequality following from the continuity of y (see also Sect. 3.6 below). □

3.6 Proof of Theorem 1

As we have mentioned before, we will take the inverse Laplace transform of $\hat{\psi}$ and deform the contour, collecting contributions from the poles in the process. We first provide the necessary estimates.

Lemma 3.22 *Assume $\text{supp}\psi_0 \in [-M, M]$, then $\sqrt{p}f(x, p)$, where $f(x, p)$ is as defined in Sect. 2, is analytic in \sqrt{p} with a square root branch at zero. Moreover,*

$$f(x, p) = \frac{\psi_0(x)}{p} + O(p^{-3/2}) + O(p^{-3/2}e^{Mi^{3/2}\sqrt{p}})$$

for large $|p|$.

Proof By integration by parts we have

$$\begin{aligned} f(x, p) &= \frac{\psi_0(x)}{2p} - \frac{e^{-i^{3/2}\sqrt{p}x}}{2p} \int_{+\infty}^x e^{i^{3/2}\sqrt{p}s} \psi'_0(s) ds \\ &+ \frac{\psi_0(x)}{2p} - \frac{e^{i^{3/2}\sqrt{p}x}}{2p} \int_{-\infty}^x e^{-i^{3/2}\sqrt{p}s} \psi'_0(s) ds \\ &= \frac{\psi_0(x)}{p} - \frac{\psi'_0(x)}{2i^{3/2}p^{3/2}} + \frac{e^{-i^{3/2}\sqrt{p}x}}{2i^{3/2}p^{3/2}} \int_{+\infty}^x e^{i^{3/2}\sqrt{p}s} \psi''_0(s) ds \\ &+ \frac{\psi'_0(x)}{2i^{3/2}p^{3/2}} - \frac{e^{i^{3/2}\sqrt{p}x}}{2i^{3/2}p^{3/2}} \int_{-\infty}^x e^{-i^{3/2}\sqrt{p}s} \psi''_0(s) ds \\ &= \frac{\psi_0(x)}{p} + \frac{i^{-3/2}}{2p^{3/2}} \int_{+\infty}^0 e^{i^{3/2}\sqrt{p}u} \psi''_0(u+x) du - \frac{i^{-3/2}}{2p^{3/2}} \int_{-\infty}^0 e^{-i^{3/2}\sqrt{p}u} \psi''_0(u+x) du. \end{aligned}$$

The lemma then follows. □

Lemma 3.23 $\tilde{\psi}(p)$ satisfies

(1) For any compact region $\Omega_1 \in \mathbb{C}$ which does not contain any pole of $\tilde{\psi}(p)$, we have

$$\sup_{p \in \Omega_1} \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) |\tilde{\psi}(p + n\omega i)|^2 < \infty.$$

In particular,

$$\sup_{p \in \Omega_1} \sum_{n=-\infty}^{\infty} |\tilde{\psi}(p + n\omega i)| < \infty.$$

(2) For any $c \geq 0$, $\int_{c-i\infty}^{c+i\infty} |\tilde{\psi}(p)| dp < \infty$.

(3) For $|\operatorname{Re}(p)| > (2r + 1)^2$ we have

$$\tilde{\psi}(p) = p^{-1/2} O(f(0, p)) = O(p^{-3/2}) + O\left(p^{-2} e^{Mi^{3/2} \sqrt{p}}\right).$$

Note that the $p^{-1/2}$ behavior of $\tilde{\psi}(p)$ near the origin does not affect the nature of these estimates, so we omit further discussions of that special case.

Proof (1) Recall that $\tilde{\psi}(i + n\omega i + z) = y_n(z)$ and that $\mathbf{y} \in \mathcal{H}$, i.e.

$$\|\mathbf{y}\|^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) |y_n|^2 < \infty.$$

Since \mathbf{y} is continuous in z on the Riemann surface of the square root, so is $\|\mathbf{y}\|$. Compactness of Ω_1 then implies $\sup_{p \in \Omega_1} \|\mathbf{y}\| < \infty$, from which the first part follows. The second part follows from the Cauchy-Schwarz inequality

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \sup_{p \in \Omega_1} |\tilde{\psi}(p + n\omega i)| \\ &= \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2})^{-1/2} (1 + |n|^{3/2})^{1/2} \sup_{p \in \Omega_1} |\tilde{\psi}(p + n\omega i)| \\ &\leq \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2})^{-1} \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) \sup_{p \in \Omega_1} |\tilde{\psi}(p + n\omega i)|^2 < \infty. \end{aligned}$$

(2) Note that by Fubini’s theorem and Cauchy-Schwarz inequality (cf. part (1)) we have

$$\begin{aligned} \int_{c-i\infty}^{c+i\infty} |\tilde{\psi}(p)| dp &= \sum_{n=-\infty}^{\infty} \int_0^1 |\tilde{\psi}(c + n\omega i + si)| ds \\ &= \int_0^1 \sum_{n=-\infty}^{\infty} |\tilde{\psi}(c + n\omega i + si)| ds \\ &\leq \sup_{p \in [c-si, c+si]} \sum_{n=-\infty}^{\infty} |\tilde{\psi}(p + n\omega i)| < \infty. \end{aligned}$$

(3) The conclusion follows from Lemma 3.2 and Lemma 3.22. □

Proposition 3.24 $\psi(x, t) = \frac{1}{2\pi i} \int_{C_1} e^{i^{3/2} \sqrt{p}|x| + pt} \tilde{\psi}(p) dp + \frac{1}{2\pi i} \int_{C_2} e^{pt} f(x, p) dp$, where the contours $C_{1,2}$ are as shown in Figs. 1 and 2. In the process of deforming the first contour, we collect contributions from the poles and we slightly rotate the branch cut by a small angle θ if a pole sits on the usual branch cut.

Fig. 1 Contour C_1 , which consists of infinitely many horizontal and vertical line segments surrounding the branch cuts (dotted lines) and poles ("x" marks). C_1 is deformed from the original contour (vertical line to the right of the imaginary axis) of the Bromwich integral

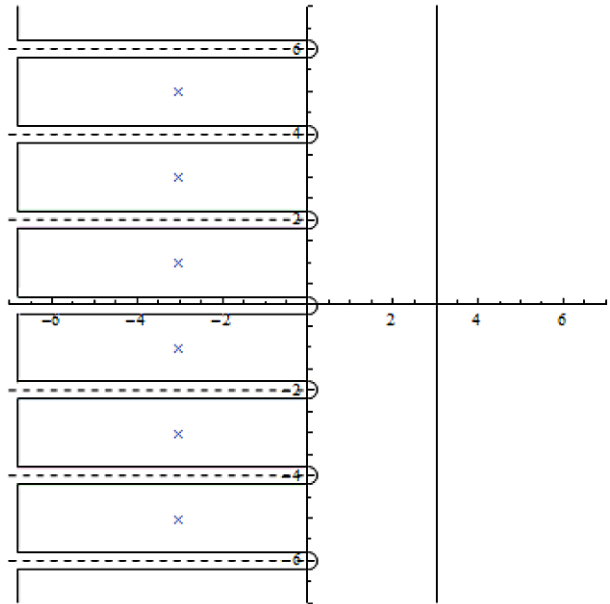
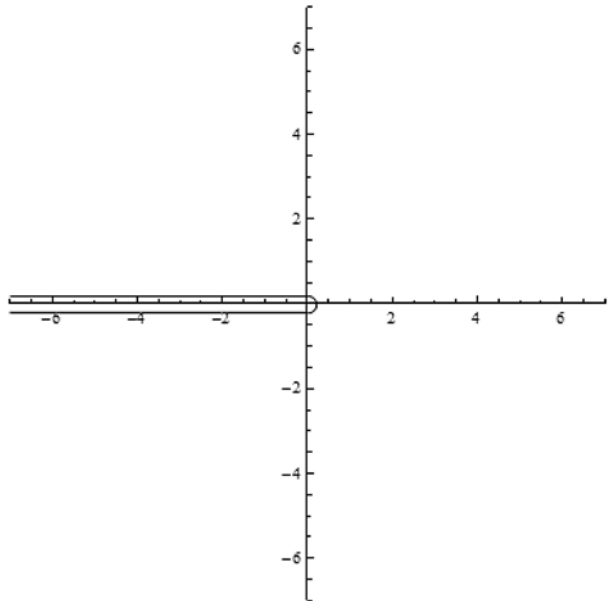


Fig. 2 Contour C_2



Proof We first note that

$$\sup_{\text{Im}(p) \geq 0} \left| e^{i^{3/2} \sqrt{p}|x|} \right| = 1$$

and

$$\sup_{\text{Im}(p) < 0, s \in \mathbb{R}} \left| e^{i^{3/2} \sqrt{p+is}|x|} \right| \leq \sup_{v \in \mathbb{R}} \left| e^{i^{3/2} \sqrt{-\text{Im}(p)(-1+iv)}|x|} \right| = e^{c_1|x|\sqrt{-\text{Im}(p)}}$$

where $c_1 = \sup_{v \in \mathbb{R}} \text{Re}(i^{3/2} \sqrt{(-1+iv)}) < \infty$.

Now, by the Bromwich integral formula

$$\begin{aligned} \psi(x, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \hat{\psi}(x, p) dp \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{i^{3/2} \sqrt{p}|x|+pt} \tilde{\psi}(p) dp + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} f(x, p) dp. \end{aligned}$$

By Lemma 3.22 we have

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} f(x, p) dp \\ &= \frac{\psi_0(x)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{p} dp + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \left(f(x, p) - \frac{\psi_0(x)}{p} \right) dp \\ &= \psi_0(x) + \frac{1}{2\pi i} \int_{C_2} e^{pt} \left(f(x, p) - \frac{\psi_0(x)}{p} \right) dp = \frac{1}{2\pi i} \int_{C_2} e^{pt} f(x, p) dp. \end{aligned}$$

As for the first contour, we only need to show that (along both sides of the branch cuts)

$$\sum_{n=-\infty}^{\infty} \int_0^{-qe^{i\theta}} e^{i^{3/2} \sqrt{s+n\omega i}|x|+st+n\omega it} \tilde{\psi}(s+n\omega i) ds < \infty,$$

$$\sum_{n=-\infty}^{\infty} \int_{-qe^{i\theta}+n\omega i}^{-qe^{i\theta}+(n+1)\omega i} e^{i^{3/2} \sqrt{p}|x|+pt} \tilde{\psi}(p) dp < \infty$$

and if the resonance is visible with the usual (or slightly rotated) branch cut, then

$$\sum_{n=-\infty}^{\infty} |A_{k,n}| < \infty.$$

The first two estimates follow from Lemma 3.23, since

$$\begin{aligned} &\left| \sum_{n=-\infty}^{\infty} \int_0^{-qe^{i\theta}} e^{i^{3/2} \sqrt{s+n\omega i}|x|+st+n\omega it} \tilde{\psi}(s+n\omega i) ds \right| \\ &\leq \left(\sup_{p \in [0, -qe^{i\theta}]} \sum_{n=-\infty}^{\infty} |\tilde{\psi}(p+n\omega i)| \right) \int_0^{-qe^{i\theta}} e^{c_1|x|\sqrt{|s|}+st} ds < \infty \end{aligned}$$

and

$$\left| \sum_{n=-\infty}^{\infty} \int_{-qe^{i\theta}+n\omega i}^{-qe^{i\theta}+(n+1)\omega i} e^{i^{3/2} \sqrt{p}|x|+pt} \tilde{\psi}(p) dp \right|$$

$$\begin{aligned} &\leq \sum_{n=-\infty}^{\infty} \int_0^{\omega i} \left| e^{i^{3/2} \sqrt{-q e^{i\theta} + n\omega i + s}|x| - q e^{i\theta} t} \tilde{\psi}(-q e^{i\theta} + n\omega i + s) \right| ds \\ &\leq e^{c_1|x|\sqrt{|q|-q} \cos \theta t} \left(\sup_{p \in [0, \omega i]} \sum_{n=-\infty}^{\infty} |\tilde{\psi}(p + n\omega i)| \right) < \infty. \end{aligned}$$

The estimates for the resonances follows from Proposition 3.21 and the Cauchy-Schwarz inequality. Since $\mathbf{A}_k \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |A_{k,n}| &= \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2})^{-1/2} (1 + |n|^{3/2})^{1/2} |A_{k,n}| \\ &\leq \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2})^{-1} \sum_{n=-\infty}^{\infty} (1 + |n|^{3/2}) |A_{k,n}|^2 < \infty. \end{aligned} \quad \square$$

Corollary 3.25 For $t > 0$, we may further deform the contour C_1 to C_3 by pushing the vertical lines left to infinity.

Proof Note that, in the proof of the previous proposition, $\int_0^{-q e^{i\theta}} e^{c_1|x|\sqrt{|s|+st}} ds$ is bounded in $\text{Re}(q) > 0$ and $e^{c_1|x|\sqrt{|q|-q} \cos \theta t} \rightarrow 0$ as $\text{Re}(q) \rightarrow \infty$.

Thus we conclude the proof of Theorem 2.1 by taking the differences between the upper and lower branches to deform the contour integrals into line integrals. To be exact, if we denote $F_s(x, \sqrt{p}) = f(x, p)$, $\tilde{\varphi}_n(\sqrt{p - n\omega i}) = \hat{\psi}(0, p - n\omega i)$, then we take $F(x, p) = F_s(x, \sqrt{p}) - F_s(x, -\sqrt{p})$ and $\varphi(p) = \hat{\psi}_n(x, \sqrt{p}) - \hat{\psi}_n(x, -\sqrt{p})$.

The last part the theorem follows immediately from Watson’s Lemma, since F and φ are clearly analytic in \sqrt{p} and has sub-exponential growth as $\text{Im}(p) \rightarrow -\infty$ (see Lemmas 3.22 and 3.23). Note also that $\tilde{\psi}(p) \sim -(1 + 2r)f(0, p)$ as $p \rightarrow 0$.

Corollary 2.1 follows from a direct calculation using (1) and (2). □

4 Further Discussion and Numerical Results

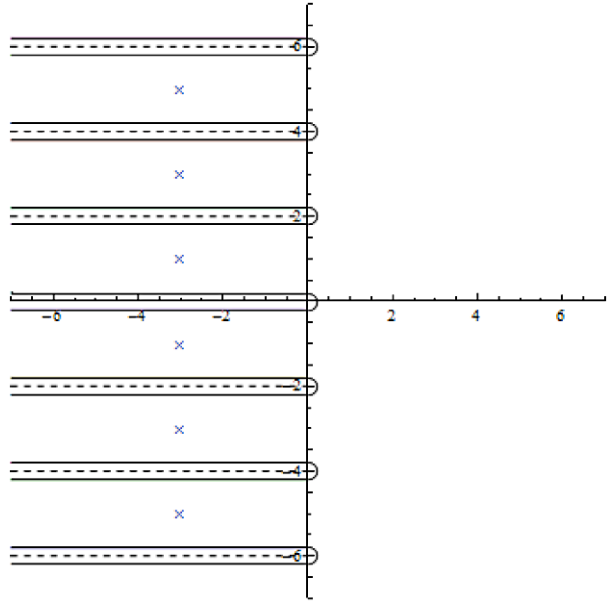
In this section we study the physical meaning of the resonances, calculate the positions of the resonances numerically, and discuss the delta potential barrier.

4.1 Metastable States and Multiphoton Ionization

When a resonance is close to but not on the imaginary axis, it corresponds to a metastable state of the wave function (see [27]). If $|x|$ is not too large, for a moderately long time the wave function is governed by the Gamow vector terms whose resonances are closest to the imaginary axis. Thus, for a fixed initial wave function, the real part of these resonances approximately measure the rate of ionization, that is, the integral of $|\psi|^2$ over a fixed spacial interval as a function of t .

It has been observed (see [5]) that the rate of ionization changes rapidly when ω is approximately equal to an integer fraction of the bound state energy (in our case, $\omega = 1/m, m \in \mathbb{N}$). This phenomenon is related to multiphoton ionization (see [5, 11–13] and

Fig. 3 Contour C_3



the references therein), a process in which an electron escapes from the nucleus by absorbing multiple photons at the same time. Since, as we mentioned in the last paragraph, the ionization rate can be measured by the position of resonances, we expect a rapid change in the real part of the resonance λ_1 when ω is near $1/m$ and r is small.

Proposition 4.1 For $\frac{1}{m+1} < \omega \leq \frac{1}{m}$, the real part of the resonance is of order r^{2m+2} for small r .

Sketch of Proof Recalling Proposition 3.13, we have $z \sim \frac{2i}{(1+\omega)^{1/2}-1} + \frac{2i}{1-(1-\omega)^{1/2}}$.

It can be shown by induction that $(\mathcal{T}_2^k \mathbf{v})_{-1}$ is a function of $h_{-1}, h_{-2}, \dots, h_{-\lfloor \frac{k}{2} \rfloor - 1}$ and of order r^{k+1} (the operators $\mathcal{T}_{1,2}$ are as defined in Proposition 3.9). Moreover, $(\mathcal{T}_2^{2k+1} \mathbf{v})_{-1} = 0$ and $(\mathcal{T}_2^{2k} \mathbf{v})_{-1} \neq 0$.

For example,

$$(\mathcal{T}_2^2 \mathbf{v})_{-1} = \frac{r^3}{h_{-1}^2 h_{-2}},$$

$$(\mathcal{T}_2^4 \mathbf{v})_{-1} = \frac{r^5 (h_{-1} + h_{-3})}{h_{-1}^3 h_{-2}^2 h_{-3}}.$$

With the notation $z = (\frac{2i}{(1+\omega)^{1/2}-1} + \frac{2i}{1-(1-\omega)^{1/2}} + \sigma)r^2$, we have

$$h_n = \sqrt{1 + n\omega + \left(\frac{2}{(1+\omega)^{1/2}-1} + \frac{2}{1-(1-\omega)^{1/2}} - i\sigma \right) r^2 - 1}$$

and $\text{Im}(\frac{h_0}{r}) = -\frac{r}{2} \text{Re}(\sigma)(1 + o(1))$. Also note that for $-m \leq n < 0$, we have $\text{Re}(h_n) \sim \sqrt{1+n\omega} - 1$ and $\text{Im}(h_n) = O(r^2 \text{Re}(\sigma))$, while for $n < -m$ we have $h_n \sim -1 +$

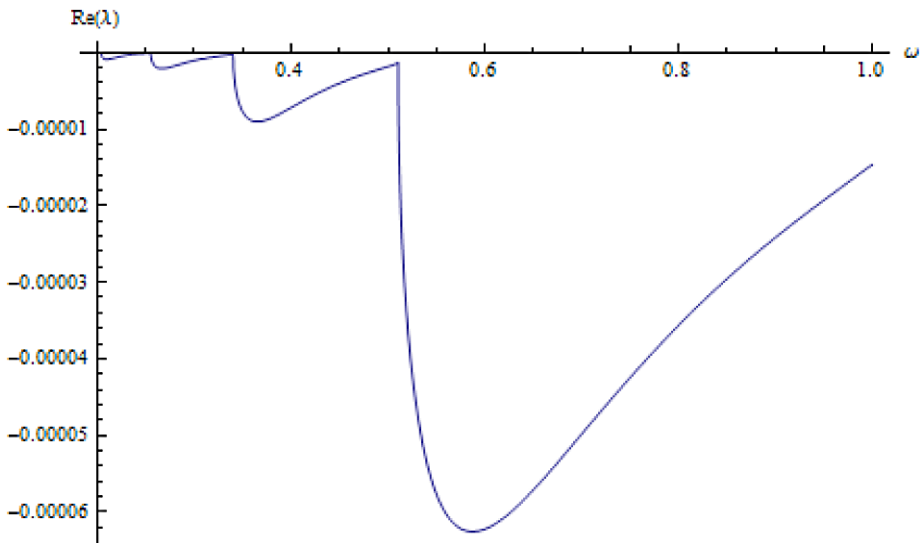


Fig. 4 Real part of the resonance as a function of ω

$\sqrt{-1 - n\omega i}$. Therefore $\text{Im}(W) = -\frac{r}{2}\text{Re}(\sigma)(1 + o(1)) - c_r r^{2m+1}(1 + O(r))$. Thus we must have $\text{Re}(\sigma) \sim \text{const.}r^{2m}$.

In particular, for $\frac{1}{3} < \omega \leq \frac{1}{2}$ we have

$$\text{Re}(\sigma) \sim \frac{2\sqrt{3\omega - 1}r^4}{3\omega(\sqrt{1 - \omega} - 1)^2(\sqrt{1 - 2\omega} - 1)^2}. \quad \square$$

Remark 4.2 In fact, it can be shown by induction that for $\frac{1}{m+1} < \omega \leq \frac{1}{m}$ we have (see also [5])

$$\text{Re}(\sigma) \sim \frac{2\sqrt{(m+1)\omega - 1}r^{2m}}{(m+1)\omega \prod_{k=1}^m (\sqrt{1 - k\omega} - 1)^2}.$$

The above proposition implies that there is indeed a rapid change in the real part of the resonance. Here we confirm this result with numerical calculations (see Fig. 4) and omit further details of the proof.

4.2 Position of Resonance: Numerical Results

As we have shown in Sect. 3.4, for small r there is only one resonance in the left half complex plane, for all choices of branch. This is, however, not always the case for general r .

We demonstrate the position of resonances in the left half plane by numerically calculating zeros of W for different r . In the graph below we show zeros of W plotted with different r and choices of branch, with $\omega = 2$. The positions of the roots are found by approximating W with sufficiently many contractive iterations and using the “FindRoot” function in *Mathematica*. The result and accuracy have been checked by (1) increasing the number of iterations (the position of the roots remains the same) and (2) visually verifying the position of the roots using density plots of (good approximations of) W .

Based on these numerical results (mainly the positions of resonances plotted in Fig. 5), we make the following observations:

- (1) Number of resonances
 - (a) For some values of r , such as those between 0.69 and 0.77, there is no visible resonance with the usual choice of branch. In other words, the Gamow vector term in Theorem 1 is absent. This happens since the resonance that was on the principal Riemann sheet for $0 < r < 0.69$ moved to the non-principal sheet (“ \times ” marks) as r increased beyond 0.69, while the “new” resonance (“+” marks) will not move to the principal sheet until r becomes as large as 0.77.
 - (b) New resonances emerge as r becomes larger. (For instance, when r is near 0.78, the “old” resonance (“ \times ” marks) has moved to the non-principal Riemann sheet, while a new one is on the principal sheet.) They can only be “born” from the imaginary axis, according to Proposition 3.19.
 - (c) With any given r , there does not seem to be more than one resonance visible with the usual choice of branch. In other words, a “new” resonance will not move into the principal Riemann sheet before the “old” resonance moves away. Note that (5) combines resonances from different r values. In fact, there is only one dot mark (indicating a resonance on the principal Riemann sheet) for each r .
- (2) Position of resonances with respect to r
 - (a) Resonances always move upward with increasing r . In other words, the imaginary part of the position of a given resonance is an increasing function of r .
 - (b) On the principal Riemann sheet, resonances move first away from the imaginary axis then towards it. New resonances move faster than older ones. These are obvious

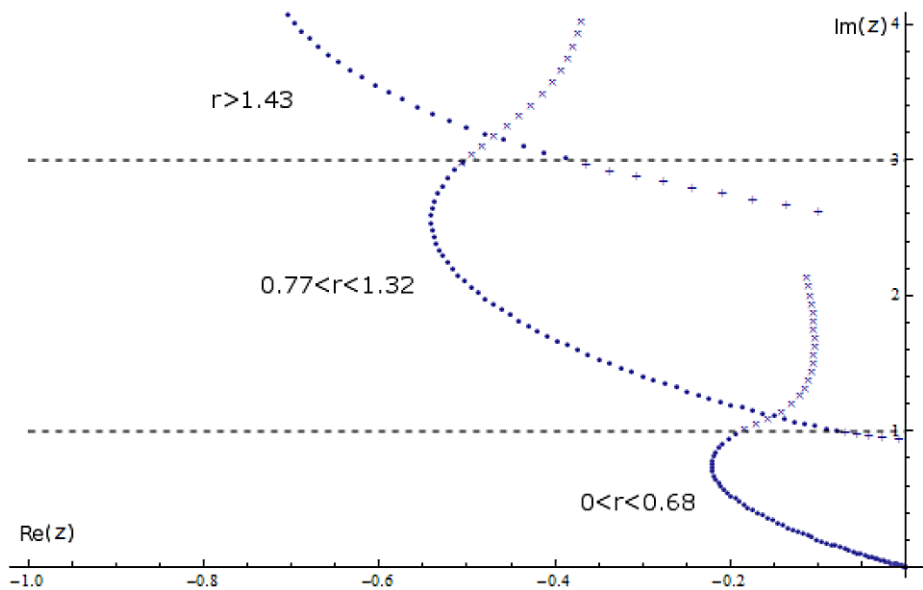


Fig. 5 Position of resonances for different r . Dots are resonances for the usual branch, and “ \times ” and “+” are those resonances continuing on the Riemann surface (they are not visible with the usual branch cut). The “ \times ” and “+” curves in the middle are on different Riemann sheets. Note that we did not plot more “ \times ” marks beyond $-0.11 + 2.1i$ due to difficulty in numerical calculation. The resonance will continue to move on the Riemann surface according to Lemma 3.18

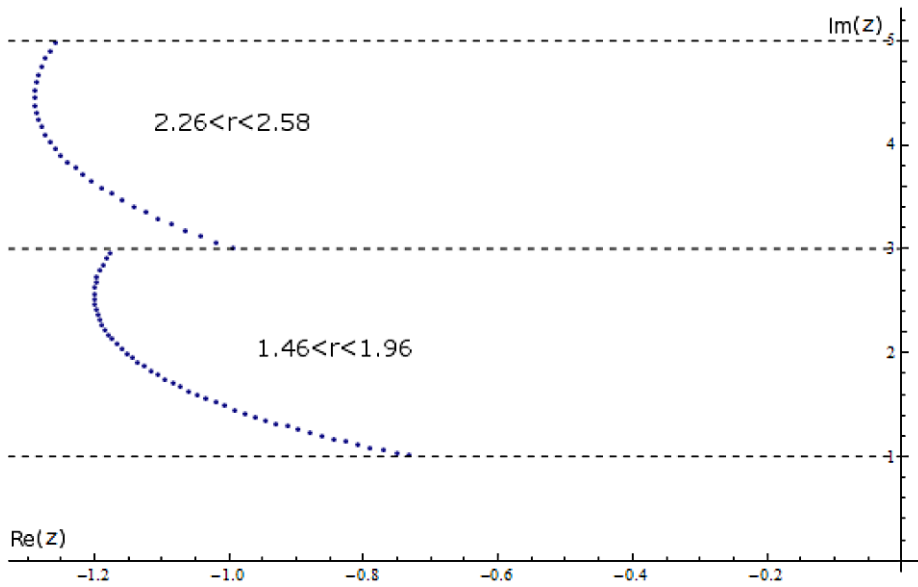


Fig. 6 Position of resonances for different r for the delta potential barrier

from Fig. 5. The step size for plotting is fixed to be 0.01, thus denser dotted curves correspond to slower variation of positions.

- (c) “Old” resonances do not move arbitrarily close to the imaginary axis with increasing r . Therefore, it is likely that the number of total resonances on the Riemann surface increases as r becomes larger.

4.3 Delta Potential Barrier

Finally, we briefly discuss the case for the delta potential barrier. The corresponding recurrence relation (see (9)) is

$$(\sqrt{-i}\sqrt{p} + 1) \hat{\psi}(0, p) = r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) + \sqrt{-i}\sqrt{p}f(0, p).$$

With a change of branch $\sqrt{p} \rightarrow -\sqrt{p}$ and changes of variables $r \rightarrow -r, f \rightarrow -f$, the above equation becomes

$$(\sqrt{-i}\sqrt{p} - 1) \hat{\psi}(0, p) = r\hat{\psi}(0, p - i\omega) + r\hat{\psi}(0, p + i\omega) + \sqrt{-i}\sqrt{p}f(0, p)$$

which is identical to (9).

Therefore essentially all the theoretical results hold for this case as well. Note, however, that for small r there is no resonance with the usual choice of branch (which corresponds to a different choice of branch in the potential barrier case, see Proposition 3.13).

For larger r , we expect the behavior of the wave function to be qualitatively similar to that with a delta potential well, since the contribution from the time-independent part will be relatively insignificant compared to the time-dependent part. This is confirmed with the graph below plotted for different r and $\omega = 2$. We choose the usual branch for simplicity.

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References

1. Gamow, G.: Zur Quantentheorie des Atomkernes. *Z. Phys.* **51**, 204–212 (1928)
2. Garrido, P., Goldstein, S., Lukkarinen, J., Tumulka, R.: Paradoxical reflection in quantum mechanics. [arXiv:0808.0610](https://arxiv.org/abs/0808.0610)
3. Skibsted, E.: Truncated Gamow functions, α -decay and the exponential law. *Commun. Math. Phys.* **104**(4), 591–604 (1986)
4. Wells, J.C., Simbotin, I., Gavrilu, M.: Multistate high-frequency Floquet theory. *Laser Phys.* **7**(3), 525–536 (1997)
5. Costin, O., Lebowitz, J.L., Ronhlenno, A.: On the complete ionization of a periodically perturbed quantum system. *CRM Proc. Lect. Notes* **27**, 51–61 (2001)
6. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics*. Academic Press, New York (1972)
7. Costin, O., Costin, R.D., Lebowitz, J.L.: Time asymptotics of the Schrödinger wave function in time-periodic potentials. *J. Stat. Phys.* **1–4**, 283–310 (2004). Special issue dedicated to Elliott Lieb on the occasion of his 70th birthday
8. Costin, O.: *Asymptotics and Borel Summability*. CRC Press, Boca Raton (2008)
9. de la Madrid, R., Gadella, M.: A Pedestrian introduction to Gamow vectors. *Am. J. Phys.* **70**(6), 626–638 (2002)
10. Kapuścik, E., Szczeszek, P.: The physical mechanism of formation of quantum mechanical Gamow states. *Found. Phys. Lett.* **18**(6), 573–580
11. Chin, S.L., Lambropoulos, P. (eds.): *Multiphoton Ionization of Atoms*. Academic Press, New York (1984)
12. Mainfray, G., Manus, G.: Multiphoton ionization of atoms. *Rep. Prog. Phys.* **54**, 1333–1372 (1991). doi:[10.1088/0034-4885/54/10/002](https://doi.org/10.1088/0034-4885/54/10/002)
13. Gribakin, G.F., Kuchiev, M.Yu.: Multiphoton detachment of electrons from negative ions. *Phys. Rev. A* **55**(5), 3760–3771 (1997)
14. Bambusi, D., Graffi, S.: Time Quasi-periodic unbounded perturbations of the Schrödinger operators and KAM methods. *Commun. Math. Phys.* **219**(2), 465–480 (2001)
15. Costin, O., Huang, M., Qiu, Z.: Ionization in damped time-harmonic fields. *J. Phys. A, Math. Theor.* **42**, 325202 (2009)
16. Costin, O., Soffer, A.: Resonance theory for Schrödinger operators. *Commun. Math. Phys.* **224**, 133–152 (2001)
17. Gesztesy, F., et al., Spectral theory and mathematical physics: a festschrift in honor of Barry Simon's 60th birthday. In: *Proceedings of Symposia in Pure Mathematics*, vol. 76.1 (2007)
18. Simon, B.: Resonances and complex scaling: a rigorous overview. *Int. J. Quantum Chem.* **14**, 529–542 (1978)
19. Soffer, A., Weinstein, M.I.: Time dependent resonance theory. *Geom. Funct. Anal.* **8**(6), 1086–1128 (1998)
20. Antoniou, I., Melnikov, Yu., Yarevsky, E.: The connection between the rigged Hilbert space and the complex scaling approaches for resonances. The Friedrichs model. *Chaos Solitons Fractals* **12**(14–15), 2683–2688 (2001)
21. Costin, O., Kruskal, M.D.: On optimal truncation of divergent series solutions of nonlinear differential systems; Berry smoothing. *Proc. R. Soc. Lond. A* **455**, 1931–1956 (1999)
22. Graffi, S., Grecchi, V., Simon, B.: Borel summability: application to the anharmonic oscillator. *Phys. Lett. B* **32**, 631–634 (1970)
23. Graffi, S., Grecchi, V.: Existence and Borel summability of resonances in hydrogen stark effect. *Lett. Math. Phys.* **2**(4), 335–341 (1978)
24. Chirka, E.M.: *Complex Analytic Sets*. Springer, New York (1989)
25. Skibsted, E.: Truncated Gamow functions, α -decay and the exponential law. *Commun. Math. Phys.* **104**(4), 591–604 (1986)
26. Costin, O., Lebowitz, J.L., Stucchio, C.: Ionization in a 1-dimensional dipole model. *Rev. Math. Phys.* **20**(7), 835–872 (2008)
27. Costin, O., Huang, M.: Gamow vectors and Borel summability. Submitted